InverseJacobiSD

Notations

Traditional name

Inverse of the Jacobi elliptic function sd

Traditional notation

$\mathrm{sd}^{-1}(z \mid m)$

Mathematica StandardForm notation

`InverseJacobiSD[z, m]`

Primary definition

09.47.02.0001.01

$$z = \mathrm{sd}(w \mid m) \; /; \; w = \mathrm{sd}^{-1}(z \mid m)$$

09.47.02.0002.01

$$\mathrm{sd}^{-1}(z \mid m) = \int_0^z \frac{1}{\sqrt{m\,t^2+1}\,\sqrt{1-(1-m)\,t^2}}\,dt \; /; \; z \in \mathbb{R} \bigwedge m\,z^2 > -1 \bigwedge (1-m)\,z^2 < 1$$

Specific values

Specialized values

For fixed *z*

09.47.03.0001.01

$$\mathrm{sd}^{-1}(z \mid 0) = \sin^{-1}(z)$$

09.47.03.0002.01

$$\mathrm{sd}^{-1}\left(z \,\middle|\, \frac{1}{2}\right) = -i\sqrt{2}\,F\left(i\sinh^{-1}\left(\frac{z}{\sqrt{2}}\right) \,\middle|\, -1\right) /; \; z > -1$$

09.47.03.0003.01

$$\mathrm{sd}^{-1}(z \mid 1) = \sinh^{-1}(z)$$

For fixed *m*

09.47.03.0004.01

$$\mathrm{sd}^{-1}(-1 \mid m) = \frac{i}{\sqrt{m}}\,F\left(i\sinh^{-1}\left(\sqrt{m}\right) \,\middle|\, \frac{m-1}{m}\right) /; \; m > 0$$

09.47.03.0005.01

$$\mathrm{sd}^{-1}\left(-\frac{1}{2} \middle| m\right) = \frac{i}{\sqrt{m}} F\left(i \sinh^{-1}\left(\frac{\sqrt{m}}{2}\right) \middle| \frac{m-1}{m}\right) /; m > 0$$

09.47.03.0006.01

$$\mathrm{sd}^{-1}(0 \mid m) = 0$$

09.47.03.0007.01

$$\mathrm{sd}^{-1}\left(\frac{1}{2} \middle| m\right) = -\frac{i}{\sqrt{m}} F\left(i \sinh^{-1}\left(\frac{\sqrt{m}}{2}\right) \middle| \frac{m-1}{m}\right) /; m > 0$$

09.47.03.0008.01

$$\mathrm{sd}^{-1}(1 \mid m) = -\frac{i}{\sqrt{m}} F\left(i \sinh^{-1}\left(\sqrt{m}\right) \middle| \frac{m-1}{m}\right) /; m > 0$$

09.47.03.0009.01

$$\mathrm{sd}^{-1}(i \mid m) = \frac{i}{\sqrt{m}} F\left(\sin^{-1}\left(\sqrt{m}\right) \middle| \frac{m-1}{m}\right) /; m < 1$$

09.47.03.0010.01

$$\mathrm{sd}^{-1}(-i \mid m) = -\frac{i}{\sqrt{m}} F\left(\sin^{-1}\left(\sqrt{m}\right) \middle| \frac{m-1}{m}\right) /; m < 1$$

Values at infinities

09.47.03.0011.01

$$\mathrm{sd}^{-1}(z \mid \infty) = 0$$

09.47.03.0012.01

$$\mathrm{sd}^{-1}(z \mid -\infty) = 0$$

09.47.03.0013.01

$$\mathrm{sd}^{-1}(\infty \mid m) = \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) /; m > 1$$

09.47.03.0014.01

$$\mathrm{sd}^{-1}(-\infty \mid m) = -\frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) /; m > 1$$

General characteristics

Domain and analyticity

$\mathrm{sd}^{-1}(z \mid m)$ is an analytical function of z and m which is defined over $\mathbb{C}^2$.

09.47.04.0001.01

$$(z * m) \longrightarrow \mathrm{sd}^{-1}(z \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

09.47.04.0002.01

$$\operatorname{sd}^{-1}(\bar{z} \mid \bar{m}) == \overline{\operatorname{sd}^{-1}(z \mid m)}$$

Quasi-reflection symmetry

09.47.04.0003.01

$$\operatorname{sd}^{-1}(-z \mid m) == -\operatorname{sd}^{-1}(z \mid m)$$

Poles and essential singularities

With respect to *m*

The function $\operatorname{sd}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to m.

09.47.04.0004.01

$$\mathcal{Sing}_m\left(\operatorname{sd}^{-1}(z \mid m)\right) == \{\}$$

With respect to *z*

The function $\operatorname{sd}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to z.

09.47.04.0005.01

$$\mathcal{Sing}_z\left(\operatorname{sd}^{-1}(z \mid m)\right) == \{\}$$

Branch points

With respect to *m*

For fixed z, the function $\operatorname{sd}^{-1}(z \mid m)$ has three branch points: $m == -\frac{1}{z^2}$, $m == \frac{z^2-1}{z^2}$, $m = \tilde{\infty}$.

09.47.04.0006.01

$$\mathcal{BP}_m\left(\operatorname{sd}^{-1}(z \mid m)\right) == \left\{-\frac{1}{z^2}, \frac{z^2-1}{z^2}, \tilde{\infty}\right\}$$

09.47.04.0007.01

$$\mathcal{R}_m\left(\operatorname{sd}^{-1}(z \mid m), -\frac{1}{z^2}\right) == 2$$

09.47.04.0008.01

$$\mathcal{R}_m\left(\operatorname{sd}^{-1}(z \mid m), \frac{z^2-1}{z^2}\right) == 2$$

09.47.04.0009.01

$$\mathcal{R}_m\left(\operatorname{sd}^{-1}(z \mid m), \tilde{\infty}\right) == \log$$

With respect to *z*

For fixed m, the function $\operatorname{sd}^{-1}(z \mid m)$ has five branch points: $z = \pm \frac{1}{\sqrt{-m}}$, $z == \pm \frac{1}{\sqrt{1-m}}$, $z = \tilde{\infty}$.

09.47.04.0010.01

$$\mathcal{BP}_z\left(\operatorname{sd}^{-1}(z \mid m)\right) == \left\{\frac{1}{\sqrt{-m}}, -\frac{1}{\sqrt{-m}}, \frac{1}{\sqrt{1-m}}, -\frac{1}{\sqrt{1-m}}, \tilde{\infty}\right\}$$

09.47.04.0011.01

$$\mathcal{R}_z\left(\mathrm{sd}^{-1}(z \mid m), \frac{1}{\sqrt{-m}}\right) = 2$$

09.47.04.0012.01

$$\mathcal{R}_z\left(\mathrm{sd}^{-1}(z \mid m), -\frac{1}{\sqrt{-m}}\right) = 2$$

09.47.04.0013.01

$$\mathcal{R}_z\left(\mathrm{sd}^{-1}(z \mid m), \frac{1}{\sqrt{1-m}}\right) = 2$$

09.47.04.0014.01

$$\mathcal{R}_z\left(\mathrm{sd}^{-1}(z \mid m), -\frac{1}{\sqrt{1-m}}\right) = 2$$

09.47.04.0015.01

$$\mathcal{R}_z\left(\mathrm{sd}^{-1}(z \mid m), \tilde{\infty}\right) = \log$$

Branch cuts

Branch cut locations: complicated

Series representations

Generalized power series

Expansions at $z = 0$

09.47.06.0001.02

$$\mathrm{sd}^{-1}(z \mid m) \propto z + \frac{1-2m}{6} z^3 + \frac{3-8m+8m^2}{40} z^5 - \ldots \;/;\; (z \to 0)$$

09.47.06.0002.01

$$\mathrm{sd}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{(1-m)^k \left(\frac{1}{2}\right)_k}{(2k+1)\, k!}\, {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2}-k; \frac{m}{m-1}\right) z^{2k+1} \;/;\; \left|(1-m)\, z^2\right| < 1$$

09.47.06.0011.01

$$\mathrm{sd}^{-1}(z \mid m) \propto z\left(1 + O\left(z^2\right)\right)$$

Expansions at $m = 0$

09.47.06.0003.01

$$\mathrm{sd}^{-1}(z \mid m) \propto \sin^{-1}(z) + \frac{z\sqrt{1-z^2}\left(z^2+1\right) + \left(z^2-1\right)\sin^{-1}(z)}{4\left(z^2-1\right)} m - \frac{z\sqrt{1-z^2}\left(6z^6 - 11z^4 - 12z^2 + 9\right) - 9\left(z^2-1\right)^2 \sin^{-1}(z)}{64\left(z^2-1\right)^2} m^2 + \ldots \;/;\; (m \to 0)$$

09.47.06.0012.01

$$\operatorname{sd}^{-1}(z\,|\,m) = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{1}{2}\right)_j z^{2j+1}}{(2j+1)\, j!}\, {}_2F_1\!\left(\frac{1}{2}, j+\frac{1}{2}; j+\frac{3}{2}; (1-m)\, z^2\right) m^j$$

09.47.06.0004.01

$$\operatorname{sd}^{-1}(z\,|\,m) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^j \left(\frac{1}{2}\right)_{j-k} \left(\frac{1}{2}\right)_k}{(2j+1)\,(j-k)!\, k!}\, z^{2j+1}\, {}_2F_1\!\left(j+\frac{1}{2}, k+\frac{1}{2}; j+\frac{3}{2}; z^2\right) m^j$$

09.47.06.0005.01

$$\operatorname{sd}^{-1}(z\,|\,m) = \sin^{-1}(z) + \frac{z\sqrt{1-z^2}\,\left(z^2+1\right) + \left(z^2-1\right)\sin^{-1}(z)}{4\left(z^2-1\right)}\, m - \frac{z\sqrt{1-z^2}\,\left(6z^6-11z^4-12z^2+9\right) - 9\left(z^2-1\right)^2 \sin^{-1}(z)}{64\left(z^2-1\right)^2}\, m^2 + \ldots$$

09.47.06.0006.01

$$\operatorname{sd}^{-1}(z\,|\,m) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^j z^{2j+2k+1} (-k)_l}{(2j+2k+1)\, j!\, k!\, l!} \left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_k m^{j+l}$$

09.47.06.0007.01

$$\operatorname{sd}^{-1}(z\,|\,m) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{z^{2j+2k+1} (1-m)^k (-m)^j}{(2j+2k+1)\, j!\, k!} \left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_k$$

09.47.06.0008.01

$$\operatorname{sd}^{-1}(z\,|\,m) = z\, F_{1\times 0\times 0}^{1\times 1\times 1}\!\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \\ \frac{3}{2};;; \end{matrix}\; (1-m)\, z^2, -m\, z^2\right)$$

09.47.06.0009.01

$$\operatorname{sd}^{-1}(z\,|\,m) = \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^j z^{2j+2r+1}\, \Gamma\!\left(j-k+\frac{1}{2}\right) \Gamma\!\left(k+r+\frac{1}{2}\right) m^j}{\pi\,(2j+2r+1)\,(j-k)!\, r!\, k!}$$

09.47.06.0010.01

$$\operatorname{sd}^{-1}(z\,|\,m) = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{1}{2}\right)_j^2 z^{2j+1}}{\left(\frac{3}{2}\right)_j\, j!}\, F_{0\times 1\times 1}^{1\times 1\times 1}\!\left(\begin{matrix} \frac{1}{2}; -j; \frac{1}{2}+j; \\ ; \frac{1}{2}-j; \frac{3}{2}+j; \end{matrix}\; 1, z^2\right) m^j$$

09.47.06.0013.01

$$\operatorname{sd}^{-1}(z\,|\,m) \propto \sin^{-1}(z)\,(1+O(m))$$

Integral representations

On the real axis

Of the direct function

09.47.07.0001.01

$$\operatorname{sd}^{-1}(z\,|\,m) = \int_0^z \frac{1}{\sqrt{m\,t^2+1}\,\sqrt{1-(1-m)\,t^2}}\, dt \,/;\, z \in \mathbb{R} \bigwedge m\, z^2 > -1 \bigwedge (1-m)\, z^2 < 1$$

09.47.07.0002.01

$$\operatorname{sd}^{-1}(z \mid m) = \frac{\sqrt{m z^2 + 1}\ \operatorname{cn}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)}{\sqrt{(m-1) z^2 + 1}} \int_0^z \frac{1}{\sqrt{m t^2 + 1}\ \sqrt{1 - (1-m) t^2}}\, dt \;/;$$
$$\neg\, \exists_{\tau,\{\tau \in \mathbb{R}, 0 < \tau < 1\}} \left(\operatorname{Im}\left(m z^2 \tau^2 + 1\right) = 0 \bigwedge m z^2 \tau^2 + 1 < 0 \bigwedge \operatorname{Im}\left(1 - (1-m) \tau^2 z^2\right) = 0 \bigwedge 1 - (1-m) \tau^2 z^2 < 0\right)$$

09.47.07.0003.01

$$\operatorname{sd}^{-1}(z \mid m) = \frac{\sqrt{m z^2 + 1}\ \operatorname{cn}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)}{\sqrt{(m-1) z^2 + 1}} \int_{z_0}^z \frac{1}{\sqrt{m t^2 + 1}\ \sqrt{1 - (1-m) t^2}}\, dt + \operatorname{sd}^{-1}(z_0 \mid m) \;/;$$
$$\neg\, \exists_{\tau,\{\tau \in \mathbb{R}, 0 < \tau < 1\}} \left(\operatorname{Im}\left(m (\tau (z - z_0) + z_0)^2 + 1\right) = 0 \bigwedge m (\tau (z - z_0) + z_0)^2 + 1 < 0 \bigwedge \right.$$
$$\left. \operatorname{Im}\left(1 - (1-m) (\tau (z - z_0) + z_0)^2\right) = 0 \bigwedge 1 - (1-m) (\tau (z - z_0) + z_0)^2 < 0\right)$$

Differential equations

Ordinary nonlinear differential equations

09.47.13.0001.01

$$w''(z) - \left(2 (1-m) m z^2 - 2 m + 1\right) z\, w'(z)^3 == 0 \;/;\; w(z) == \operatorname{sd}^{-1}(z \mid m)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.47.16.0001.01

$$\operatorname{sd}^{-1}(-z \mid m) == -\operatorname{sd}^{-1}(z \mid m)$$

Identities

Functional identities

09.47.17.0001.01

$$\left((m-1) m z_1^2 z_2^2 - 1\right)^2 \operatorname{sd}(w(z_1) + w(z_2) \mid m)^4 -$$
$$2 \left(\left(\left((m-1)\left(z_1^2 + z_2^2\right) m + 4 m - 2\right) z_2^2 + 1\right) z_1^2 + z_2^2\right) \operatorname{sd}(w(z_1) + w(z_2) \mid m)^2 + \left(z_1^2 - z_2^2\right)^2 == 0 \;/;\; w(z) == \operatorname{sd}^{-1}(z \mid m)$$

Differentiation

Low-order differentiation

With respect to z

09.47.20.0001.02

$$\frac{\partial \operatorname{sd}^{-1}(z \mid m)}{\partial z} == \frac{\operatorname{cn}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)}{(m-1) z^2 + 1}$$

09.47.20.0002.01

$$\frac{\partial \operatorname{sd}^{-1}(z \mid m)}{\partial z} = \frac{1}{\sqrt{m z^2 + 1} \sqrt{1 - (1-m) z^2}} \;/;\; z \in \mathbb{R} \bigwedge m z^2 > -1 \bigwedge (1-m) z^2 < 1$$

09.47.20.0003.02

$$\frac{\partial^2 \operatorname{sd}^{-1}(z \mid m)}{\partial z^2} = -\frac{z \left(2 (m-1) m z^2 + 2 m - 1\right) \operatorname{cn}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)}{\left((m-1) z^2 + 1\right)^2 \left(m z^2 + 1\right)}$$

09.47.20.0011.01

$$\frac{\partial^2 \operatorname{sd}^{-1}(z \mid m)}{\partial z^2} = \frac{\sqrt{m z^2 + 1} \operatorname{cn}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)}{\sqrt{(m-1) z^2 + 1}} \frac{\partial \frac{1}{\sqrt{m z^2+1} \sqrt{1-(1-m) z^2}}}{\partial z}$$

With respect to *m*

09.47.20.0004.02

$$\frac{\partial \operatorname{sd}^{-1}(z \mid m)}{\partial m} = -\frac{E\left(\operatorname{am}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right) \mid m\right) + (m-1) \operatorname{sd}^{-1}(z \mid m) - \frac{m z \operatorname{nc}\left(\operatorname{sd}^{-1}(z|m)|m\right)}{m z^2+1}}{2 (m-1) m}$$

09.47.20.0005.01

$$\frac{\partial \operatorname{sd}^{-1}(z \mid m)}{\partial m} = \frac{1}{2 \sqrt{m-1}\, m} \left(i F\left(i \sinh^{-1}\left(\sqrt{m-1}\, z\right) \,\middle|\, \frac{m}{m-1}\right) - \frac{\sqrt{m-1}\, m z^3}{\sqrt{(m-1) z^2 + 1} \sqrt{m z^2 + 1}} - i E\left(i \sinh^{-1}\left(\sqrt{m-1}\, z\right) \,\middle|\, \frac{m}{m-1}\right)\right) /;$$

$$z \in \mathbb{R} \bigwedge m z^2 > -1 \bigwedge (1-m) z^2 < 1$$

09.47.20.0006.02

$$\frac{\partial^2 \operatorname{sd}^{-1}(z \mid m)}{\partial m^2} = \frac{1}{4 (m-1)^2 m^2} \Bigg(3 \operatorname{sd}^{-1}(z \mid m) (m-1)^2 + F\left(\operatorname{am}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right) \mid m\right) (m-1) + (4 m - 2) E\left(\operatorname{am}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right) \mid m\right) + \frac{1}{\left((m-1) z^2 + 1\right)^2} \Bigg(m \operatorname{cn}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right) \Bigg(z^2 \left((m-1)^2 z^2 + m - 1\right) \sqrt{\frac{1}{m z^2 + 1}} \operatorname{sn}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right) - \frac{z^2 \left(z^2 + m \left((m-1)(5 m - 2) z^4 + (8 m - 7) z^2 + 3\right) - 1\right) \operatorname{ds}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)}{m z^2 + 1} \Bigg)\Bigg)\Bigg)$$

09.47.20.0012.01

$$\frac{\partial^3 \mathrm{sd}^{-1}(z \mid m)}{\partial m^3} = \frac{1}{8\,(m-1)^3\,m^3} \Bigg((-23\,(m-1)\,m-8)\,E\!\left(\mathrm{am}\!\left(\mathrm{sd}^{-1}(z \mid m) \mid m\right) \mid m\right) - (m-1)\,(11\,m-7)\,F\!\left(\mathrm{am}\!\left(\mathrm{sd}^{-1}(z \mid m) \mid m\right) \mid m\right) + \frac{1}{\left((m-1)\,z^2+1\right)^3 \left(m\,z^2+1\right)^2} \Bigg(m\,\mathrm{cn}\!\left(\mathrm{sd}^{-1}(z \mid m) \mid m\right) \Bigg(\left(m\,z^2+1\right) \Big(\left((m-1)^2\,m^2\,(m\,(53\,m-36)+7)\,z^8 + 2\,(m-1)\,m\,(m\,(m\,(86\,m-109)+43)-6)\,z^6 + \left(m\left(m\left(206\,m^2-376\,m+231\right)-58\right)+5\right) z^4 + 2\,(m\,(m\,(54\,m-73)+32)-5)\,z^2 + 21\,m^2-18\,m+5\right) \mathrm{dn}\!\left(\mathrm{sd}^{-1}(z \mid m) \mid m\right) - (m-1)\left((m-1)\,z^2+1\right) \sqrt{\frac{1}{m\,z^2+1}} \left((m-1)\,m\,(11\,m-7)\,z^6 + (m\,(17\,m-18)+4)\,z^4 + (5\,m-3)\,z^2 - 1\right) \Big) \mathrm{sn}\!\left(\mathrm{sd}^{-1}(z \mid m) \mid m\right) - (m-1)\,m\,z\left((m-1)\,m\,(m\,(20\,m-9)-2)\,z^8 + \left(m\left(66\,m^2-80\,m+23\right)-1\right) z^6 + \left(78\,m^2-69\,m+13\right) z^4 + 2\,(19\,m-9)\,z^2+6\right) \Bigg) - 15\left((m-1)^2\,z^2+m-1\right)^3 \left(m\,z^2+1\right)^2 \mathrm{sd}^{-1}(z \mid m) \Bigg) \Bigg)$$

Symbolic differentiation

With respect to z

09.47.20.0013.01

$$\frac{\partial^n \mathrm{sd}^{-1}(z \mid m)}{\partial z^n} == \mathrm{sd}^{-1}(z \mid m)\,\delta_n + \frac{\mathrm{cn}\!\left(\mathrm{sd}^{-1}(z \mid m) \mid m\right)}{(m-1)\,z^2+1} \sum_{j=0}^{n-1} \frac{(-1)^j\,m^j\,(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2\,z)^{-2\,j+n-1}} \sum_{k=0}^{j} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} \left(\frac{m-1}{m}\right)^{j-k} \left(m\,z^2+1\right)^{-k} \left(1-(1-m)\,z^2\right)^{k-j} \;/;\; n \in \mathbb{N}$$

09.47.20.0014.01

$$\frac{\partial^n \mathrm{sd}^{-1}(z \mid m)}{\partial z^n} == \mathrm{sd}^{-1}(z \mid m)\,\delta_n + \frac{\mathrm{cn}\!\left(\mathrm{sd}^{-1}(z \mid m) \mid m\right)}{(m-1)\,z^2+1} \sum_{j=0}^{n-1} \frac{(-1)^j\,2^{2\,j-n+1}\,m^j\,z^{2\,j-n+1} \left(1-(1-m)\,z^2\right)^{-j} \left(\frac{1}{2}\right)_j (1-n)_{2(n-j)-2}}{(n-j-1)!} \left(\frac{m-1}{m}\right)^j {}_2F_1\!\left(\frac{1}{2}, -j; \frac{1}{2}-j; \frac{m\left(m\,z^2-z^2+1\right)}{(m-1)\left(m\,z^2+1\right)}\right) \;/;\; n \in \mathbb{N}$$

09.47.20.0015.01

$$\frac{\partial^n \mathrm{sd}^{-1}(z \mid m)}{\partial z^n} == \mathrm{sd}^{-1}(z \mid m)\,\delta_n + \frac{\sqrt{m\,z^2+1}\;\mathrm{cn}\!\left(\mathrm{sd}^{-1}(z \mid m) \mid m\right)}{\sqrt{(m-1)\,z^2+1}} \frac{\partial^{n-1} \frac{1}{\sqrt{m\,z^2+1}\,\sqrt{1-(1-m)\,z^2}}}{\partial z^{n-1}} \;/;\; n \in \mathbb{N}^+$$

09.47.20.0007.01

$$\frac{\partial^n \operatorname{sd}^{-1}(z \mid m)}{\partial z^n} = \frac{2^{n-1} \pi z^{n-1} (n-1)! \operatorname{cn}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)}{(m-1) z^2 + 1} \sum_{j=0}^{n-1} \frac{(m-1)^{n-j-1} m^j \left((m-1) z^2 + 1\right)^{j-n+1} \left(m z^2 + 1\right)^{-j}}{j! (n-j-1)! \Gamma\left(\frac{1}{2} - j\right) \Gamma\left(j - n + \frac{3}{2}\right)}$$
$$ {}_2F_1\left(\frac{1-j}{2}, -\frac{j}{2}; \frac{1}{2} - j; 1 + \frac{1}{m z^2}\right) {}_2F_1\left(\frac{j-n+2}{2}, \frac{j-n+1}{2}; j - n + \frac{3}{2}; 1 + \frac{1}{(m-1) z^2}\right) /; n \in \mathbb{N}^+$$

With respect to *m*

09.47.20.0008.02

$$\frac{\partial^n \operatorname{sd}^{-1}(z \mid m)}{\partial m^n} = \frac{z^{2n+1}}{2n+1} \frac{\sqrt{m z^2 + 1} \operatorname{cn}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)}{\sqrt{(m-1) z^2 + 1}}$$
$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2} - k\right)_k \left(k - n + \frac{1}{2}\right)_{n-k} F_1\left(n + \frac{1}{2}; \frac{1}{2} - k + n, k + \frac{1}{2}; n + \frac{3}{2}; (1-m) z^2, -m z^2\right) /; n \in \mathbb{N}$$

09.47.20.0016.01

$$\frac{\partial^n \operatorname{sd}^{-1}(z \mid m)}{\partial m^n} = \frac{\sqrt{m z^2 + 1} \operatorname{cn}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)}{\sqrt{(m-1) z^2 + 1}} \frac{\partial^n \frac{F\left(\sin^{-1}\left(\sqrt{1-m}\, z\right) \middle| \frac{m}{m-1}\right)}{\sqrt{1-m}}}{\partial m^n} /; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

09.47.20.0009.01

$$\frac{\partial^\alpha \operatorname{sd}^{-1}(z \mid m)}{\partial z^\alpha} = z^{1-\alpha} \sqrt{\pi}\, \tilde{F}_{2\times0\times0}^{2\times1\times1}\left(\begin{matrix} \frac{1}{2}, 1; \frac{1}{2}; \frac{1}{2}; \\ \frac{3-\alpha}{2}, 1 - \frac{\alpha}{2};;; \end{matrix} -m z^2, (1-m) z^2\right) /; z \in \mathbb{R} \bigwedge (1-m) z^2 < 1$$

With respect to *m*

09.47.20.0010.01

$$\frac{\partial^\alpha \operatorname{sd}^{-1}(z \mid m)}{\partial m^\alpha} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_j \frac{(j+k)! (-1)^{j+k} z^{2j+2k+1} m^{j+k-\alpha}}{(2j + 2k + 1) \Gamma(j + k - \alpha + 1) k! j!} {}_2F_1\left(\frac{1}{2} + j + k, j + \frac{1}{2}; \frac{3}{2} + j + k; z^2\right) /;$$
$$-1 < z < 1 \land -1 < m < 1$$

Integration

Indefinite integration

Involving only one direct function

09.47.21.0001.01

$$\int \operatorname{sd}^{-1}(z \mid m) \, dz = \operatorname{sd}^{-1}(z \mid m) z - \frac{1}{\sqrt{m-1} \sqrt{m}} \log\left(\frac{\operatorname{cd}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)}{\sqrt{m-1}} + \frac{\operatorname{nd}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)}{\sqrt{m}}\right)$$

Involving only one direct function with respect to *m*

09.47.21.0002.01

$$\int \operatorname{sd}^{-1}(z \mid m)\, dm = 2\sqrt{m-1}\, i\left(E\left(i\sinh^{-1}\left(\sqrt{m-1}\, z\right) \,\middle|\, \frac{m}{m-1}\right) - F\left(i\sinh^{-1}\left(\sqrt{m-1}\, z\right) \,\middle|\, \frac{m}{m-1}\right)\right) + \frac{1}{z\sqrt{(m-1)z^2+1}} \left(2(m-1)\sqrt{m z^2+1}\, z^2 - \sqrt{(m-1)z^2+1}\, \log\left(\frac{1}{4}\left((2m-1)z^2 + 2\sqrt{(m-1)z^2+1}\sqrt{m z^2+1} + 2\right)\right) - 2\sqrt{(m-1)z^2+1} + 2\sqrt{m z^2+1}\right) /; z > 0 \land m > 0$$

Representations through more general functions

Through hypergeometric functions of two variables

09.47.26.0001.01

$$\operatorname{sd}^{-1}(z \mid m) = z\, F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \\ \frac{3}{2};;; \end{matrix}\; (1-m)z^2, -m z^2\right)$$

09.47.26.0002.01

$$\operatorname{sd}^{-1}(z \mid m) = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{1}{2}\right)_j^2 z^{2j+1}}{\left(\frac{3}{2}\right)_j j!} F_{0\times 1\times 1}^{1\times 1\times 1}\left(\begin{matrix} \frac{1}{2}; -j; \frac{1}{2}+j; \\ ; \frac{1}{2}-j; \frac{3}{2}+j; \end{matrix}\; 1, z^2\right) m^j$$

Through other functions

Involving some hypergeometric-type functions

09.47.26.0003.01

$$\operatorname{sd}^{-1}(z \mid m) = z\, F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -m z^2, (1-m)z^2\right) /; z \in \mathbb{R} \bigwedge (1-m)z^2 < 1$$

Representations through equivalent functions

With inverse function

09.47.27.0001.01

$$\operatorname{sd}\left(\operatorname{sd}^{-1}(z \mid m) \,\middle|\, m\right) = z$$

With related functions

Involving cd^{-1}

09.47.27.0002.01

$$\operatorname{sd}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \left(K\left(\frac{m}{m-1}\right) - \operatorname{cd}^{-1}\left(\sqrt{1-m}\, z \,\middle|\, \frac{m}{m-1}\right)\right) /; z \in \mathbb{R} \land m > 1$$

Involving cn^{-1}

09.47.27.0003.01

$$\operatorname{sc}^{-1}(z \mid m) = -i \operatorname{cn}^{-1}\left(\sqrt{z^2+1} \,\middle|\, 1-m\right) /; 0 < z < 1 \land 0 < m < 1$$

Involving cs^{-1}

09.47.27.0004.01

$$\text{sc}^{-1}(z \mid m) = \text{cs}^{-1}\left(\frac{1}{z} \middle| m\right) /; z > 0 \land m \in \mathbb{R}$$

Involving dc^{-1}

09.47.27.0005.01

$$\text{sc}^{-1}(z \mid m) = i\left(\frac{1}{\sqrt{1-m}} \text{dc}^{-1}\left(i\, z \middle| \frac{1}{1-m}\right) - K(1-m)\right) /; z \in \mathbb{R} \land 0 < m < 1$$

Involving dn^{-1}

09.47.27.0006.01

$$\text{sc}^{-1}(z \mid m) = i\left(\frac{1}{\sqrt{m-1}} \text{dn}^{-1}\left(i\, z \middle| \frac{m}{m-1}\right) - K(1-m)\right) /; -1 < z < 1 \land m > 1$$

Involving ds^{-1}

09.47.27.0007.01

$$\text{sd}^{-1}(z \mid m) = \text{ds}^{-1}\left(\frac{1}{z} \middle| m\right) /; z > 0 \land m > 1$$

Involving nc^{-1}

09.47.27.0008.01

$$\text{sc}^{-1}(z \mid m) = i\, K(1-m) - \frac{1}{\sqrt{m}} \text{nc}^{-1}\left(z \middle| \frac{1}{m}\right) /; -1 < z < 1 \land m > 1$$

Involving nd^{-1}

09.47.27.0009.01

$$\text{sd}^{-1}(z \mid m) = \frac{1}{\sqrt{m}}\left(\text{nd}^{-1}\left(i\, z \sqrt{m} \middle| \frac{1}{m}\right) - i\, K\left(1 - \frac{1}{m}\right)\right) /; z \in \mathbb{R} \land m > 1$$

Involving ns^{-1}

09.47.27.0010.01

$$\text{sc}^{-1}(z \mid m) = -i\, \text{ns}^{-1}\left(-\frac{i}{z} \middle| 1-m\right) /; z > 0 \land m \in \mathbb{R}$$

Involving sc^{-1}

09.47.27.0011.01

$$\text{sc}^{-1}(z \mid m) = -i\, \text{sn}^{-1}(i\, z \mid 1-m)$$

Involving sn^{-1}

09.47.27.0012.01

$$\text{sd}^{-1}(z \mid m) = -\frac{i}{\sqrt{m}} \text{sn}^{-1}\left(\sqrt{-m}\, z \middle| \frac{m-1}{m}\right) /; -1 < z < 1 \land m > 0$$

Involving elliptic integrals

09.47.27.0013.01

$$\mathrm{sd}^{-1}(z \mid m) = -\frac{i}{\sqrt{m}} F\left(i \sinh^{-1}\left(\sqrt{m}\, z\right) \,\middle|\, \frac{m-1}{m}\right) /; |z| < 1 \land |m| < 1$$

09.47.27.0015.01

$$\mathrm{sd}^{-1}(z \mid m) = \frac{\sqrt{m z^2 + 1}\, \mathrm{cn}\left(\mathrm{sd}^{-1}(z \mid m) \mid m\right)}{\sqrt{1-m}\, \sqrt{(m-1) z^2 + 1}} F\left(\sin^{-1}\left(\sqrt{1-m}\, z\right) \,\middle|\, \frac{m}{m-1}\right) /;$$

$$\neg \exists_{\tau,\{\tau \in \mathbb{R}, 0<\tau<1\}} \left(\mathrm{Im}\left(m z^2 \tau^2 + 1\right) = 0 \bigwedge m z^2 \tau^2 + 1 < 0 \bigwedge \mathrm{Im}\left(1 - (1-m) \tau^2 z^2\right) = 0 \bigwedge 1 - (1-m) \tau^2 z^2 < 0\right)$$

09.47.27.0016.01

$$\mathrm{sd}^{-1}(z \mid m) = \mathrm{sd}^{-1}(z_0 \mid m) + \frac{\sqrt{m z^2 + 1}\, \mathrm{cn}\left(\mathrm{sd}^{-1}(z \mid m) \mid m\right)}{\sqrt{1-m}\, \sqrt{(m-1) z^2 + 1}} \left(F\left(\sin^{-1}\left(\sqrt{1-m}\, z\right) \,\middle|\, \frac{m}{m-1}\right) - F\left(\sin^{-1}\left(\sqrt{1-m}\, z_0\right) \,\middle|\, \frac{m}{m-1}\right)\right) /;$$

$$\neg \exists_{\tau,\{\tau \in \mathbb{R}, 0<\tau<1\}} \left(\mathrm{Im}\left(m (\tau (z - z_0) + z_0)^2 + 1\right) = 0 \bigwedge m (\tau (z - z_0) + z_0)^2 + 1 < 0 \bigwedge \right.$$
$$\left. \mathrm{Im}\left(1 - (1-m) (\tau (z - z_0) + z_0)^2\right) = 0 \bigwedge 1 - (1-m) (\tau (z - z_0) + z_0)^2 < 0\right)$$

Involving other related functions

09.47.27.0014.01

$$\mathrm{sd}^{-1}(z \mid m) = -\frac{\sqrt{z_2^2}}{z_2} \mathrm{elog}(z_1, z_2; a, b) /; \{a, b, z_1\} = \left\{2 m - 1, m (m-1), \frac{1}{z^2}\right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge z > 0 \bigwedge m > 1$$

History

– N. H. Abel (1826)
– A. G. Greenhill (1892)
– L. M. Milne–Thompson (1948)

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